Optimum and Practical Noncausal Smoothing Filters for Estimating Carrier Phase With Phase Process Noise

W. J. Hurd
Communications Ground Systems Section

Noncausal smoothing filters have significant performance advantages over causal filters for estimation of the phase of radio frequency signals. This article presents the optimum smoothing filters for the cases where the spectral density of the carrier phase is of the form $f^{-\alpha}$, *where* $\alpha = 3$ *is typical of oscillator noise and* $\alpha = 8/3$ *is typical of phase perturbations due to propagation through charged particles, such as solar wind. It is shown that the optimum smoothers perform approximately 5-dB better than causal filters of optimum bandwidth, and that the performance of practical-to-implement smoothing filters is within 0.03 to 0.24 dB of the optimum smoothers. Some important examples where smoothing can be used are in the Deep Space Network Galileo Telemetry System, where the data are recorded and processed in nonreal time; in the DSN Block V Receiver, which provides data buffering of sufficient size to enable near-real-time smoothing; and in processing recorded radio science data, where carrier amplitude and phase are the primary data.*

I. Introduction

The performance of deep-space communications links is often limited by the ability to estimate the phase of the radio frequency carrier. Noncausal smoothing filters have significant performance advantages over phase-locked loops (PLLs) or other causal filters for estimating carrier phase when the estimates can be made in nonreal time or near-real time, rather than in real time. An important opportunity for nonreal-time smoothing is the Deep Space Network Galileo Telemetry System, in which the data are recorded and then processed. A second example is in radio science, where the carrier amplitude and phase, rather than telemetry, are the primary data of interest. A potential application of near-real-time smoothing is in the DSN Block V Receiver. This digital receiver was designed to enable storage of the data samples for sufficient time to implement smoothing estimates of carrier phase; however, the algorithms have not been implemented to date.

Previous work [1–3] has shown that use of noncausal smoothing filters to estimate carrier phase can improve the phase estimation by up to a factor of 4, or 6 dB, compared with casual filters. These works modeled the phase noise on the carrier as white velocity or white acceleration. This model was used because it is suitable for solution by Kalman filtering [1,2] or nonlinear estimation theory [3]. One disadvantage to this approach is that the models do not accurately represent typical phase process noise.

The white velocity and accelerations correspond to process spectral densities shaped as f^{-2} and f^{-4}, respectively. The dominant components of typical phase noise spectra vary as f^{-3} for oscillator noise and as $f^{-8/3}$ for phase noise dominated by propagation through charged particles, such as solar plasma. A second disadvantage is that implementation of the resulting smoothing filters is computationally intensive, and practical approximations were not presented.

In the present work, a spectral domain approach is used rather than a time domain approach. The well-known general solution [4] for minimum mean-square error, infinite delay, linear smoothing filters is applied to the actual phase noise spectra, resulting in optimum linear smoothers. Then, several smoothers of fixed form are evaluated by determining their optimum bandwidths, and the performances of these simple smoothers are compared with that of the optimum smoother. It is shown that a simple rectangular weighting-function smoother is only 0.10- to 0.24-dB worse than the optimum, for $f^{-8/3}$ and f^{-3} phase spectra, respectively. Finally, it is shown that the optimum smoother is approximately 5-dB better than typical phased-locked loops and other causal filters, even when the bandwidths of these filters are optimized.

II. The Optimum Linear Estimator Using an Infinite Delay Smoothing Filter

The optimum linear estimator using an infinite delay smoothing filter is a well-known and straightforward result of Wiener theory. Following Davenport and Root [4, Chapter 11], let the phase process we are estimating be the signal, $s(t)$, and let $n(t)$ be additive noise. Assume $s(t)$ and $n(t)$ are independent real-valued wide-sense stationary random processes. Then the minimum mean-square error infinite delay smoothing filter has transfer function

$$H_o(j2\pi f) = \frac{S_s(f)}{S_s(f) + S_n(f)} \tag{1}$$

where$S_s(f)$ and $S_n(f)$ are the spectral densities of the signal process being estimated and the additive noise. Since S_s and S_n are real-valued and even-symmetric about zero, so is H_o, and so is the corresponding weighting function, $h_o(t)$. Thus, the optimum estimator is a symmetric smoothing filter; in general, it has infinite delay.

For any linear estimator $H(j2\pi f)$, with input $s(t) + n(t)$, the spectral density of the estimation error is

$$S_e(f) = S_s(f)|1 - H(j2\pi f)|^2 + S_n(f)|H(j2\pi f)|^2 \tag{2}$$

For notational convenience, we now replace $H(j2\pi f)$ with $H(f)$. The mean-square error (for any H) can be calculated as

$$\sigma_e^2 = \sigma_s^2 + \sigma_n^2 \tag{3a}$$

where

$$\sigma_s^2 = 2\int_0^\infty S_s(f)|1 - H(f)|^2 df \tag{3b}$$

is the contribution due to the signal process and

$$\sigma_n^2 = 2\int_0^\infty S_n(f)|H(f)|^2 df \tag{3c}$$

is the contribution due to noise.

As an aside, we note that for the simple case of infinite delay and independent signal and noise, the optimum result in Eq. (1) can be derived by differentiating σ_e^2 with respect to H.

A. The Optimum Estimator for Typical Phase Noise

Phase process noise typically is dominated by low-frequency spectral components, which vary as $|f|^{-\alpha}$, where $\alpha = 3$ for typical oscillator noise and $\alpha \approx 8/3$ for propagation effects due to charged particles, such as solar plasma. For these cases, we can write

$$S_s(f) = \frac{A}{2}|f|^{-\alpha} \quad (\alpha > 0) \tag{4}$$

where $A = 2S_s(1)$ is the one-sided spectral density at 1 Hz. Now assume that the additive noise is white receiver noise, with spectral density $N_0/2$. Then, from Eq. (1),

$$H_o(f) = \frac{1}{1 + |f|^\alpha N_0/A} \quad (\alpha > 0) \tag{5}$$

B. Resulting Performance

For the optimum estimator, the noise contribution to the error is

$$\begin{aligned}\sigma_n^2 &= N_0 \int_0^\infty H_o^2(f) df \\ &= N_0 \int_0^\infty \frac{1}{[1 + f^\alpha N_0/A]^2} df \\ &= N_0 \left(\frac{A}{N_0}\right)^{1/\alpha} \int_0^\infty \frac{dx}{[1 + x^\alpha]^2}\end{aligned} \tag{6}$$

The contribution due to the phase process is

$$\sigma_s^2 = A\int_0^\infty f^{-\alpha}\left[1 - \frac{1}{1 + f^\alpha N_0/A}\right]^2 df \tag{7}$$

1. The case of $\alpha = 3$. For $\alpha = 3$, corresponding to an oscillator-noise-dominated phase spectrum, the integral in Eq. (6) evaluates to $4\pi/3^{5/2} = 0.80613$. Thus,

$$\sigma_n^2 = 0.80613 N_0 \left(\frac{A}{N_0}\right)^{1/3} \tag{8}$$

Note that this means that the optimum estimator is a smoother with a one-sided noise bandwidth of $0.80613(A/N_0)^{1/3}$ Hz.

Equation (7) also can be integrated explicitly, and results in

$$\sigma_s^2 = \frac{1}{2}\sigma_n^2 \tag{9}$$

The overall minimum mean-square error for f^{-3} phase noise is

$$\sigma_e^2 = 1.209 N_0 \left(\frac{A}{N_0}\right)^{1/3} \tag{10}$$

2. The case of $\alpha = 8/3$. For $\alpha = 8/3$, the charged-particle propagation noise case, the integrals were evaluated numerically, obtaining

$$\sigma_m^2 = 0.797 N_0 \left(\frac{A}{N_0}\right)^{3/8} \tag{11}$$

$$\sigma_s^2 = 0.478 N_0 \left(\frac{A}{N_0}\right)^{3/8} \tag{12}$$

and

$$\sigma_e^2 = 1.275 N_0 \left(\frac{A}{N_0}\right)^{3/8} \tag{13}$$

Note that, for both $\alpha = 3$ and $\alpha = 8/3$, the optimum estimator results in $\sigma_n^2 = (\alpha - 1)\sigma_s^2$. We show in the next section that this is not an accident.

III. Optimum Bandwidths for Fixed-Shape Filters

In this section, we find a simple expression for the optimum bandwidths for estimation filters of fixed shapes. We show that the overall mean-square estimation error is proportional to the optimum bandwidth, with the constant of proportionality depending only on α. We then present numerical results for some interesting smoothers and causal filters, and compare their performances with that of the optimum smoother.

A. General Solution

Let $G(f)$ be any fixed-shape filter with unity noise bandwidth, i.e.,

$$\int_0^\infty |G(f)|^2 \, df = 1 \tag{14}$$

Then the filter of the same shape with noise bandwidth b Hz has transfer function $G(f/b)$:

$$H(f) = G\left(\frac{f}{b}\right) \tag{15}$$

With the assumed phase process noise spectrum of slope $-\alpha$ and additive white noise with N_0, the mean-square phase error is

$$\sigma_e^2 = N_0 b + A \int_0^\infty f^{-\alpha} \left|1 - G\left(\frac{f}{b}\right)\right|^2 df \tag{16}$$

We write this as

$$\sigma_e^2 = N_0 b + A b^{1-\alpha} I \tag{17}$$

where

$$I = \int_0^\infty f^{-\alpha} \left|1 - G(f)\right|^2 df \tag{18}$$

depends only on the shape of the filter.

Minimizing Eq. (17), the optimum bandwidth is

$$b = \left(\frac{A}{N_0}\right)^{1/\alpha} [(\alpha - 1)I]^{1/\alpha} \tag{19}$$

The optimum bandwidth b varies as the $1/\alpha$ power of the ratio of signal spectral density to noise spectral density at 1 Hz, and depends on the filter shape through I.

At the optimum bandwidth, the process noise contribution always is $1/(\alpha - 1)$ times the additive noise contribution, i.e.,

$$\sigma_s^2 = A b^{1-\alpha} I = \frac{1}{\alpha - 1} \sigma_n^2 \tag{20}$$

and the total error is

$$\sigma_e^2 = \left(\frac{\alpha}{\alpha - 1}\right) N_0 b \tag{21}$$

B. Results for Various Smoothers and Causal Filters

Table 1 presents the optimum bandwidth b_o for several different smoothing filters and causal filters. The results are normalized to $N_0 = A = 1$. From Eq. (19), the optimum bandwidths for other values of N_0 and A are

$$b_o(A, N_0) = \left(\frac{A}{N_0}\right)^{1/\alpha} b_o \tag{22}$$

and the minimum mean-square error is given by Eq. (21).

Table 1 also shows the degradation in estimation of phase (loss in dB) using the various smoothers and causal filters as compared with the optimum smoother. It is seen that the performances of two very simple smoothers are very close to optimum; for $\alpha = 8/3$, rectangular and exponential weighting-function smoothers are only 0.10- and 0.03-dB worse than optimum. Causal filters with optimum bandwidths typically perform 5-dB worse than the optimum smoother. The best causal filter found for both $\alpha = 3$ and $\alpha = 8/3$ is the second-order critically damped filter, which is 4.90- and 4.48-dB worse than optimum for $\alpha = 3$ and $\alpha = 8/3$, respectively. The very simple rectangular weighting-function filter is 4.66- and 4.45-dB better than the best causal filter found, for $\alpha = 3$ and $\alpha = 8/3$, respectively.

Table 1. Smoothers and causal filters of fixed shape and optimum bandwidth compared with the optimum smoother for $N_0 = A = 1$.

Case	Description	Optimum bandwidth and loss			
		$\alpha = 3$		$\alpha = 8/3$	
		b_o	Loss, dB	b_o	Loss, dB
	Smoothers				
1	Optimum smoother	0.80613	0	0.797	0
2	Rectangular passband	1.0	0.94	1.0	0.99
3	Rectangular weighting function	0.8513	0.24	0.816	0.10
4	Symmetric exponential weighting	0.825	0.10	0.802	0.03
	Causal filters				
5	Rectangular weighting function	∞	∞	2.60	5.14
6	First-order (exponential weighting)	∞	∞	2.47	4.91
7	Second-order, critically damped	2.489	4.90	2.234	4.48
8	Third-order, critically damped	2.752	5.33	2.490	4.95
9	Type III phase-locked loop [5, Eqs. (6)–(8)]	2.629	5.13	2.354	4.70

IV. Implementation Considerations and Future Work

Up to this point, we have assumed that the signal (phase) process and the noise add linearly. Of course, this is not true. In a causal estimator, such as a phase-locked loop, linearity can be approached by using a predicted value of phase to maintain the phase error within the near-linear range of the phase detector. A similar approach can be used with a smoother, where the smoother is used in nonreal time to improve the estimates made by a causal filter. This works well when the causal estimator performs well enough so the phase error estimates are near linear, i.e., when the mean-square error of the PLL is less than approximately 0.2 rad^2 [3].

A Fourier transform can be used very simply to implement the rectangular weighting-function smoother. Choosing the frequency and phase that maximize the energy of the transform will result in the appropriate phase estimate for the center of the time interval. Degradations due to nonlinear phase will be minimized if variations due to dynamics are modeled or tracked.

Maximization of energy should not be done over all frequencies of a fast Fourier transform, but only over the minimum frequency range necessary to cover frequency uncertainty. This will prevent errors due to outliers. After initial acquisition, the smoother can be updated without doing the entire transform for each new data point. Instead, the new data point can be added, the oldest point deleted, and the transform values for a few closely spaced frequencies updated.

Future work needs to be done to determine the threshold performance of the smoother(s) as implemented. Phase-locked loops threshold when the mean-square error is approximately 0.2 rad^2, which corresponds to a loop signal-to-noise ratio (SNR) of 7 dB (linear model), with no process noise. We expect that the smoother implemented by the Fourier transform, with tracking of the frequency and phase, will threshold somewhere near the same mean-square error as the PLL. Since smoothers achieve the anticipated threshold mean-square error of approximately 0.2 rad^2 at an SNR of 4- to 5-dB lower than required by PLLs, we expect the threshold SNR for smoothers to be significantly better than for PLLs. This needs to be verified.

V. Conclusions

The minimum mean-square error linear estimator for a phase process with an $f^{-\alpha}$ spectrum is an infinite delay smoothing filter (where $\alpha = 3$ is typical of oscillator noise and $\alpha = 8/3$ is typical of propagation through charged particles). The optimum smoother performs 4.90- and 4.48-dB better than the best causal filter studied for $\alpha = 3$ and $\alpha = 8/3$, respectively. It is very easy to implement smoothers to perform nearly as well as optimum. In particular, a smoother with a rectangular weighting function is only 0.24- and 0.10-dB worse than optimum for $\alpha = 3$ and $\alpha = 8/3$. Thus, this simple smoother will gain 4.7 or 4.4 dB over causal filters, such as phase-locked loops, for $\alpha = 3$ and $\alpha = 8/3$.

References

[1] C. A. Pomalaza Raez and W. J. Hurd, "Improved Carrier Tracking by Smoothing Estimators," *IEEE Trans. Aerosp. Electron. Syst.*, vol. AES-21, pp. 610–618, September 1985.

[2] C. A. Pomalaza Raez and W. J. Hurd, "Carrier Tracking by Smoothing Filter Improves Symbol SNR," *IEEE Trans. Aerosp. Electron. Syst.*, vol. AES-22, no. 5, pp. 554–558, September 1986.

[3] R. Kumar and W. J. Hurd, "Fixed Lag Smoothers for Carrier Phase and Frequency Tracking," presented at the IASTED International Symposium on Applied Control and Identification, Los Angeles, California, December 10–12, 1986; also in *The Telecommunications and Data Acquisition Progress Report 42-88, October–December 1986*, Jet Propulsion Laboratory, Pasadena, California, pp. 67–78, February 15, 1987.

[4] W. B. Davenport, Jr. and W. L. Root, *An Introduction to the Theory of Random Signals and Noise*, Chapter 11, New York: McGraw Hill, 1958.

[5] V. A. Vilnrotter, W. J. Hurd, and D. H. Brown, "Optimized Tracking of RF Carriers With Phase Noise, Including Pioneer 10 Results," *The Telecommunications and Data Acquisition Progress Report 42-91, July–September 1987*, Jet Propulsion Laboratory, Pasadena, California, pp. 141–157, November 15, 1987.